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Axial flow in trailing line vortices

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A characteristic feature of a steady trailing line vortex from one side of a wing, and of other types of line vortex, is the existence of strong axial currents near the axis of symmetry. The purpose of this paper is to account in general terms for this axial flow in trailing line vortices. The link between the azimuthal and axial components of motion in a steady line vortex is provided by the pressure; the radial pressure gradient balances the centrifugal force, and any change in the azimuthal motion with distance x downstream produces an axial pressure gradient and consequently axial acceleration.

It is suggested, in a discussion of the evolution of an axisymmetric line vortex out of the vortex sheet shed from one side of a wing, that the two processes of rolling-up of the sheet and of concentration of the vorticity into a smaller cross-section should be distinguished; the former always occurs, whereas the latter seems not to be inevitable.

In § 4 there is given a similarity solution for the flow in a trailing vortex far downstream where the departure of the axial velocity from the free stream speed is small. The continual slowing-down of the azimuthal motion by viscosity leads to a positive axial pressure gradient and consequently to continual loss of axial momentum, the asymptotic variation of the axial velocity defect at the centre being as $x^{-1} \log x$.

The concept of the drag associated with the core of a trailing vortex is introduced, and the drag is expressed as an integral over a transverse plane which is independent of x. This drag is related to the arbitrary constant appearing in the above similarity solution.

1. Introduction

Steady axisymmetric flow fields in which the vorticity has large magnitude in the neighbourhood of the axis of symmetry are common in aeronautics and geophysics (e.g. trailing vortices from lifting surfaces, jet-intake vortices, bath-drain vortices, tornadoes). An interesting and prominent feature of such flow fields is that strong axial currents occur near the axis of symmetry. This is a little surprising at first sight because it often happens also that the axial pressure gradient everywhere outside the region near the axis is manifestly too small to generate such axial motions. One would like to have clear ideas about the mechanism, presumably inviscid, by which strong axial motions are generated near the axis initially, and one would like also to know if the effect of viscosity, which is likely to be significant near the axis in view of the large gradients there, modifies these axial motions in ways other than by direct frictional retardation.

Both these points can be clarified a little by a consideration of the steady trailing line vortex from one side of a wing. First we see what information can be gained from general equations for steady axisymmetric flow with vorticity confined to a region near the axis.

2. Steady axisymmetric flow in a trailing vortex

We use cylindrical co-ordinates x, r, ϕ, with corresponding velocity components u, v, w. Then the equations of motion for steady axisymmetric flow of incompressible fluid are, without approximation,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u, \tag{2.1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}-\frac{w^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 v-\frac{v}{r^2}\right), \tag{2.2}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial r}+\frac{vw}{r} = \nu\left(\nabla^2 w-\frac{w}{r^2}\right), \tag{2.3}$$

where
$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}.$$

The equation of mass conservation is satisfied identically when

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad v = -\frac{1}{r}\frac{\partial \psi}{\partial x}. \tag{2.4}$$

Other useful dependent variables are

$$C = rw,$$

representing $(2\pi)^{-1}$ times the circulation round a symmetrically placed circle, and the Bernoulli function, or (ρ^{-1} times the) total pressure,

$$H = \frac{p}{\rho}+\tfrac{1}{2}(u^2+v^2+w^2).$$

If the fluid is inviscid, C and H are functions of ψ alone, and the equation governing the flow is then (Squire 1956)

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial \psi}{\partial r} = r^2\frac{dH}{d\psi}-C\frac{dC}{d\psi}. \tag{2.5}$$

The axial component of vorticity is $r^{-1}\partial C/\partial r$.

The interest here is in flow fields in which axial gradients are of small magnitude compared with radial gradients. This allows the boundary-layer-type approximation

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial r}, \quad v \ll u.$$

The only change in equations (2.1) and (2.3) is that now

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r},$$

but (2.2) reduces to the approximate form

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{w^2}{r}. \tag{2.6}$$

We note that (2.1), which can be regarded as an equation for the axial component u, is coupled with the azimuthal component w solely through the axial variation of centrifugal force; this is the key to the understanding either of the inviscid generation of strong axial motions or their subsequent modification (indirectly) by viscosity. The pressure will be supposed to be uniform, and equal to p_0 say, far from the axis, whence (2.6) gives

$$\frac{p_0 - p}{\rho} = \int_r^\infty \frac{w^2}{r}\,dr, \quad = \int_r^\infty \frac{C^2}{r^3}\,dr. \tag{2.7}$$

We shall consider flow fields in which C tends rapidly to a constant, C_0 say, as $r \to \infty$. The region in which the vorticity is non-zero and C varies will be spoken of as the vortex 'core'.

In a simple illustrative case in which C is constant for $r > a$ and zero for $r < a$ (a cylindrical vortex sheet), we have

$$\frac{p_0 - p}{\rho} = \frac{C_0^2}{2a^2};$$

thus the pressure in the core increases if the core diameter ($2a$) increases with distance downstream, leading to axial deceleration, whereas if the core diameter decreases as x increases there is a fall in pressure and axial acceleration. (The relevance of (2.7) to the equation of motion (2.1) is in contrast to the conventional boundary-layer situation in which the pressure variation across the layer is negligibly small and has no effect on the development of the layer.) This variation of the axial velocity in the core is in the direction required by conservation of mass and the given variation of core diameter. It follows that a change in either direction—increase or decrease of the core diameter with distance downstream—is self-consistent, qualitatively at least. Both kinds of change are likely to occur in different circumstances.

In the case of flow in a trailing vortex from one side of a wing in an infinite body of fluid, all streamlines originate in a region where the pressure is uniform and equal to p_0 and the fluid velocity is uniform with components $(U, 0, 0)$. Some streamlines will have passed through the boundary layer at the wing surface, or some other region where the effect of viscous forces is appreciable, and the Bernoulli function at any point in the vortex may therefore be written as

$$\frac{p}{\rho} + \tfrac{1}{2}(u^2 + v^2 + w^2) = \frac{p_0}{\rho} + \tfrac{1}{2}U^2 - \Delta H.$$

Then, on neglecting the small term $\frac{1}{2}v^2$ and making use of (2·7), we have, for the axial velocity at any point,

$$\begin{aligned} u^2 &= U^2 - \frac{C^2}{r^2} + 2\int_r^\infty \frac{C^2}{r^3}\,dr - 2\Delta H \\ &= U^2 + \int_r^\infty \frac{1}{r^2}\frac{\partial C^2}{\partial r}\,dr - 2\Delta H. \end{aligned} \tag{2.8}$$

If the vorticity in the trailing vortex is one-signed, as is normally the case, $\partial C^2/\partial r > 0$ and the second term on the right-hand side of (2.8) is positive; this second term is the increase in u^2 which by Bernoulli's theorem would accompany the pressure drop due to centrifugal force in the absence of total head losses. Total head losses do of course occur, but there appears not to be any reason to expect a close connexion between the second and third terms on the right-hand side of (2.8) or to expect dominance of the third term, at any rate in the simpler types of trailing vortex. Consider for example the case of laminar flow at very large Reynolds number at the wing surface. The vortex sheet shed from the wing here may remain as a discrete sheet of small thickness for some distance downstream after rolling up into a spiral (Mangler & Smith 1959), with $\Delta H = 0$ outside this sheet. The circulation, and consequently also the axial velocity u, will then increase with distance from the axis in a succession of finite jumps as the different turns of the spiral sheet are crossed. At positions between adjacent turns of the sheet there is zero total head loss. The distributions of C and u are ultimately made continuous and smooth by the diffusive action of viscosity, and this process is inevitably accompanied by losses of total head throughout the vortex core, but it is not likely that the general magnitude of u is changed thereby. (In the closely related problem of a vortex in which the velocity is independent of x, but depends on t, it is possible to follow mathematically the transition from distributions of C and u with steps to continuous distributions due to the action of viscosity. The changes in the distributions of C and u are here independent and $\int_0^\infty ur\,dr$ is constant. The order of magnitude of u changes with t only inasmuch as the total width of the vortex core changes.)

Consequently it appears to be sensible to consider the magnitude of the axial velocity in the absence of total head losses, as a particular and not untypical case. The interesting feature of (2.8) is that it shows there is then an excess axial velocity, and a large one, in the core of the vortex. Outside the core, where C is constant, we have $u = U$; but inside the core $u > U$ and u increases monotonically toward the axis. Furthermore, if the azimuthal velocity w is appreciably larger in magnitude than the free stream speed U, as is often so in practice, u and w are comparable in magnitude. In the simple example in which a core of radius a rotates rigidly with angular velocity Ω, and with $\Delta H = 0$, we have

$$w = \frac{\Omega a^2}{r}, \quad u = U, \quad \text{for} \quad r \geqslant a,$$

and

$$w = \Omega r, \quad u = \{U^2 + 2\Omega^2(a^2 - r^2)\}^{\frac{1}{2}}, \quad \text{for} \quad r \geqslant a.$$

3. The development of an axisymmetric trailing vortex

Although the existence of strong axial flow has been seen to be an inevitable feature of the core of a trailing vortex, provided only that total head losses are not large, it will be useful to consider briefly the development of the excess axial velocity. There is also the associated and important question of the way in which the diameter of the vortex core is determined by the upstream conditions. The evolution of an axisymmetric and approximately cylindrical trailing vortex out of the vortex sheet shed by one side of a wing is complicated by dependence on the precise geometry of the wing, and only loose qualitative arguments seem to be possible.

It is as well to distinguish two different processes in this transition from vortex sheet near the wing to trailing vortex further downstream, viz. (1) the rolling-up of the sheet, and (2) the contraction (or expansion) of the region of non-zero vorticity in a plane transverse to the free stream. The process of rolling-up of a vortex sheet with a free edge is well understood in principle, although it is difficult to work out the details in a particular case. The free edge of the vortex sheet curls over, under the influence of the induced velocity field of the vortex sheet, and takes up the form of a spiral with a continually increasing number of turns, as depicted in figure 1, which is one of many reproductions of a sketch due to Prandtl.† Observation of the velocity distribution behind a wing supports such a picture (see, for example, *Modern Developments in Fluid Dynamics*, edited by S. Goldstein, § 12).

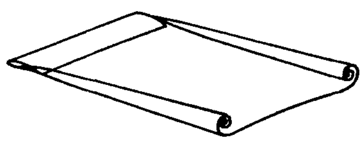

FIGURE 1. Prandtl's sketch of the rolling-up of a trailing vortex sheet.

The second of the two processes is a change in lateral dimensions of the partially rolled-up vortex. Observation of wing-tip vortices leaves no doubt that in some circumstances this change is a contraction, yielding a more concentrated vortex, although it is not clear what distinguishes these circumstances. Some authoritative writers have given the impression that the two processes are equally inevitable; for instance, we find, in § 12 of *Modern Developments*, the statement: 'The (plane) trailing vortex sheet... is not a possible stable form. It would roll up at its edges, the vorticity being concentrated more and more in the rolled-up portions, until it presented the appearance of two concentrated vortices at a distance apart somewhat less than the span of the aerofoil.' However, I believe that, whereas rolling-up of the free edge of vortex sheet is inevitable, concentra-

† Since the strength of the vortex sheet is zero downstream from the centre of the wing, one might ask why the sheet from one side of the wing does not have two free edges, both of which roll up. The vortex sheet does indeed distort over its whole area, but it is obvious from the expression for the induced velocity due to a sheet of given form that the rate of distortion of the sheet is greatest where the gradient of strength of the sheet is greatest, and this occurs near the 'edge' shed from the wing-tip.

tion of the vorticity is not, and that contraction and expansion of the cross-section are both possible in principle. The notion of concentration of the trailing vortex seems to be based on the assumption that in a two-dimensional flow a vortex sheet (which then appears as a line in the plane of flow) with a free edge rolls up like a carpet, leaving little space between successive turns of the spiral. This is erroneous; in a two-dimensional flow the continual increase in the number of turns in the spiral is accompanied by a lengthening of the line representing the sheet in the plane of flow and by an increase in the radius of curvature of the outer-most turn (until it is comparable with the initial length of the sheet), and the fluid enclosed by the outer turns of the spiral cannot be squeezed out. It is known that the dispersion or second moment

$$\sum_i \kappa_i(x_i^2+y_i^2)$$

of a finite group of point vortices of strength κ_i in inviscid fluid is an invariant of the motion (Lamb, *Hydrodynamics*, 6th edition, § 157), and it is not difficult to establish a similar result for a continuous distribution of vorticity in two dimensions (or for a distribution with discontinuities across certain lines) for which the dispersion is finite. Thus, in a two-dimensional flow, the degree of concentration of a region of one-signed vorticity, as measured by the dispersion at any rate, does not tend to increase or decrease. Other authors (Birkhoff & Fisher 1959; Hama & Burke 1960) have noticed that a tendency to concentration of a spiral vortex sheet does not appear to be consistent with the properties of two-dimensional motion.

An explanation of such 'tightening' or concentration of a trailing vortex as does occur must be sought in the three-dimensionality of the flow, or, more specifically, in the existence in the core of a non-zero and negative divergence of the velocity vector in planes transverse to the trailing vortex. We have seen that the axial velocity in the core of an axisymmetric vortex may be quite large, sometimes as much as several times the free stream speed, showing that elements of fluid entering the vortex core may experience appreciable axial acceleration. The occurrence of appreciable concentration of the trailing vortex thus depends on whether axial acceleration of these elements occurs before or after they enter the vortex core; if it occurs *after* they enter the core, the core diameter far downstream will be smaller than near the wing and a concentrated trailing vortex will result. The vortex core is of course not well defined near the wing, so that the phrase 'before or after entering the vortex core' cannot have a precise meaning. The occurrence of contraction or expansion of the vortex core in any particular case appears to depend on the configuration of the wing and the trailing vortex near the wing.

A case in which the processes of rolling-up and concentration proceed simultaneously and can be described analytically is the 'conical' flow field on one side of a slender delta wing of small aspect ratio shedding vorticity from its straight leading edge (Roy 1952). With the approximation of axi-symmetry of the flow, the velocity and pressure in the vortex are here functions of r/x alone, and vorticity of one sign is continually being fed in at the outer edge of the vortex core at the rate corresponding to increase of the circulation as x, at a given value of r/x. The axial velocity gradient $\partial u/\partial x$ thus varies as x^{-1}, at a given value of r/x, and is positive (owing to the smaller pressure inside the core than outside it), showing continual *concentration* of the vorticity in the core although at a rate which diminishes with distance downstream. The fact that the diameter of the vortex core increases as x is a little misleading; increase in size of the core and concentration of the vortex (in the sense of a non-zero rate of increase of axial vorticity with respect to x) are compatible here because vorticity is being added continually at a radius which increases with x and the vortex core is being built up by external means.

4. A similarity solution for the flow in a trailing vortex far downstream

During the rolling-up process the effect of viscosity is likely to be confined to a diffusive thickening of the vortex sheet shed by the wing. The number of convolutions of the spiral into which the sheet forms increases continually, and ultimately the neighbouring turns of a spiral are close enough for viscous spreading to make the distribution of vorticity a smooth one. This is the justification for treating the trailing vortex far downstream as axisymmetric.

Viscosity will of course continue to have an effect on the vortex after the vorticity distribution has been made continuous, and will presumably lead ultimately to a slow diffusive increase of the core diameter as $x^{\frac{1}{2}}$. It is of some interest to deduce the way in which the gradual slowing-down of the azimuthal motion by viscous action leads to an increased pressure at the axis (there being less centrifugal force then), and so to an axial deceleration of the core fluid. It proves to be possible to reveal these effects explicitly in a similarity solution which holds very far downstream, where u differs from the free stream speed U by a small amount only, as in the theory of wakes without swirl.

Far downstream, where the boundary-layer-type approximations $\partial/\partial x \ll \partial/\partial r$ and $v \ll u$ are supplemented by the approximation

$$|u-U| \ll U,$$

the equation of motion (2.1) reduces to

$$U\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{4.1}$$

(2.2) reduces to (2.6) as before, and (2.3) becomes

$$U\frac{\partial w}{\partial x} = \nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{w}{r^2}\right). \tag{4.2}$$

This latter equation can also be written as

$$U\frac{\partial C}{\partial x} = \nu r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial C}{\partial r}\right), \tag{4.3}$$

which is the same, apart from replacement of t by x/U, as the equation for viscous decay of the circulation in a two-dimensional motion. The explicit expression for p provided by (2.7) may be substituted in (4.1) to give

$$U\frac{\partial u}{\partial x} - \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \int_r^\infty \frac{1}{r^3}\frac{\partial C^2}{\partial x}dr$$

$$= \frac{2\nu}{U}\int_r^\infty \frac{C}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial C}{\partial r}\right)dr, \tag{4.4}$$

in which the right-hand side represents the axial deceleration arising from the viscous decay of the azimuthal motion.

The solution of (4.2) or (4.3) for w or C as a function of r at any value of x may be written down explicitly in terms of a given distribution of w as a function of r at some smaller value of x by making use of the known properties of the heat-conduction equation satisfied by the vorticity. As is well known, the asymptotic form of the solution as $x \to \infty$ is

$$C = rw = C_0(1 - e^{-\eta}), \tag{4.5}$$

where $$\eta = \frac{Ur^2}{4\nu x}$$

and C_0 is the (non-zero) value of C at large r. With this expression for C the pressure (see (2.7)) is given by

$$\frac{p_0 - p}{\rho} = \frac{C_0^2 U}{8\nu x}P(\eta),$$

where $$P(\eta) = \int_\eta^\infty \frac{(1-e^{-\xi})^2}{\xi^2}d\xi$$

$$= \frac{(1-e^{-\eta})^2}{\eta} + 2ei(\eta) - 2ei(2\eta), \tag{4.6}$$

and the function $$ei(\eta) = \int_\eta^\infty \frac{e^{-\xi}}{\xi}d\xi$$

has been tabulated. Equation (4.4) for u now becomes

$$U\frac{\partial u}{\partial x} - \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = -\frac{C_0^2 U}{8\nu x^2}\left(P + \eta\frac{dP}{d\eta}\right). \tag{4.7}$$

Equation (4.4) or (4.7) is of the same form as the equation for radial conduction of heat in a solid in two dimensions (again with t replaced by x/U) with a maintained and distributed heat source whose density (heat released per unit interval of x/U and of πr^2) is a function of r and x which at large values of x takes the form

$$x^{-2} \times \text{func}\,(r^2/x).$$

If the heat source density were zero for values of x above a certain value, the solution for $U - u$ would tend, as $x \to \infty$, to the form

$$\frac{BU^2}{8\nu x}\exp\left(-\frac{Ur^2}{4\nu x}\right),$$

where B is proportional to the total heat released. But the heat source density is not zero here and the nature of the asymptotic solution may be different. We obtain a clearer view of the asymptotic dependence of u on x by integrating all terms of (4.7) over a cross-sectional plane:

$$\frac{d}{dx}\int_0^\infty (U-u)\,r\,dr = \frac{C_0^2}{4Ux}\int_0^\infty \frac{d(\eta P)}{d\eta}\,d\eta,$$

$$= \frac{C_0^2}{4Ux} \tag{4.8}$$

since $P \sim \eta^{-1}$ as $\eta \to \infty$. Thus

$$\int_0^\infty (U-u)\,r\,dr = \frac{C_0^2}{4U}\log\frac{xU}{\nu} + \text{const.} \tag{4.9}$$

where ν/U has been used as a convenient unit of length in the logarithm.

This relation suggests we should look for an asymptotic solution of (4.7) of the form

$$u = U - \frac{C_0^2}{8\nu x}\log\frac{xU}{\nu}\,Q_1(\eta) + \frac{C_0^2}{8\nu x}\,Q_2(\eta) - L\frac{U^2}{8\nu x}e^{-\eta}, \tag{4.10}$$

where L is a constant with the dimensions of area, and the last term, the complementary function, accounts for any initial velocity defect which may be independent of the circulation. When (4.10) is substituted in (4.7), there are some terms proportional to $x^{-2}\log(xU/\nu)$ and the remaining terms are proportional to x^{-2}. The former group of terms have zero sum if

$$\eta Q_1'' + Q_1' + \eta Q_1' + Q_1 = 0, \tag{4.11}$$

of which the only solution free from singularities is proportional to $\exp(-\eta)$. The multiplicative constant is required by the integral relation (4.9) to be unity, whence we have

$$Q_1(\eta) = e^{-\eta}. \tag{4.12}$$

The group of terms proportional to x^{-2} have zero sum if

$$\eta Q_2'' + Q_2' + \eta Q_2' + Q_2 = -Q_1 + P + \eta P'. \tag{4.13}$$

One integration, and choice of the constant of integration to avoid a singularity at $\eta = 0$, gives

$$Q_2' + Q_2 = \frac{e^{-\eta}-1}{\eta} + P.$$

From a second integration we have

$$Q_2(\eta) = e^{-\eta}\int_0^\eta \left(\frac{1-e^{\eta}}{\eta} + P\,e^{\eta}\right)d\eta + M\,e^{-\eta}. \tag{4.14}$$

The term containing the constant of integration M can be regarded as absorbed in the last term of (4.10), which is equivalent to 'normalizing' the function Q_2 so that $Q_2(0) = 0$. In terms of tabulated functions, we have, after substituting from (4.6) and putting $M = 0$,

$$Q_2(\eta) = e^{-\eta}\{\log\eta + ei(\eta) - 0{\cdot}807\} + 2ei(\eta) - 2ei(2\eta). \tag{4.15}$$

The functions $P(\eta)$, $Q_1(\eta)$ and $Q_2(\eta)$ have been evaluated and are shown graphically in figure 2.

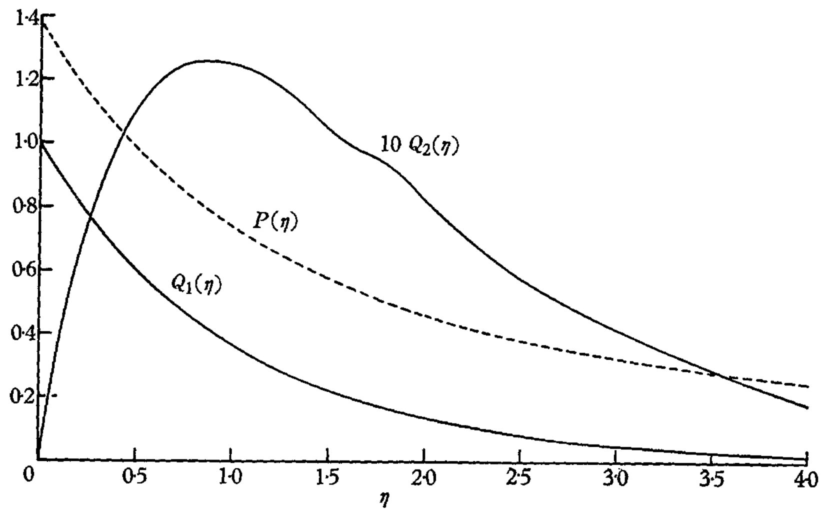

FIGURE 2. The functions P (defining the pressure, see (4.6)), Q_1 and Q_2 (defining the axial velocity, see (4.10)).

A solution of the 'similarity' form (4.10) thus exists. It is evident also that the axial velocity distribution tends to the form (4.10) as $x \to \infty$ for arbitrary conditions at some initial value of x, since the difference between u and the similarity solution (4.10) satisfies a heat-conduction equation with zero heat source density. It appears that the effect of the decay of the swirling motion is to increase the pressure in the vortex core and consequently to decrease the axial velocity there; in this way the induced drag on the wing associated with the generation of the trailing vortex is gradually manifested as an ordinary wake with an axial velocity defect. The integral over a lateral plane of the axial velocity defect arising from this conversion of azimuthal motion increases as $\log x$, but the magnitude of $U-u$ at any point is continually being diminished by diffusive spreading and so varies as $x^{-1}\log x$.

To complete the analysis we should inquire whether some restriction must be placed on the azimuthal velocity, or, equivalently, on C_0, for the above results to be consistent with the initial assumption that $|u-U| \ll U$. The maximum value of Q_2 is about 0·13, so that the axial velocity defect given by (4.10) is a small fraction of U at all values of r provided

$$\frac{C_0^2}{8\nu x}\left(\log\frac{xU}{\nu} - 0{\cdot}13\right) + L\frac{U^2}{8\nu x} \ll U. \tag{4.16}$$

It does not seem to be possible to turn this into a simple restriction on C_0, because the constant L is a resultant of contributions from the initial velocity defect (which may be independent of C_0) and from the slowing down of azimuthal motion in the pre-similarity stage; moreover, L is affected by the arbitrary choice of ν/U as the length unit in the logarithm. A more meaningful condition requires the expression of L in terms of some fundamental parameter of the flow as a whole. Such a parameter is introduced in the next section. But in any event we see that the requirement (4.16) is certainly satisfied at sufficiently large values of x.

The term on the right-hand side of (4.10) which ultimately dominates the axial velocity defect is the one proportional to $x^{-1}\log x$, giving

$$(U-u)_{\max.} \sim \frac{C_0^2}{\nu x}\log\frac{xU}{\nu}, \quad (w)_{\max.} \sim C_0\left(\frac{U}{\nu x}\right)^{\frac{1}{2}},$$

the maxima being with respect to r. However, in practice the presence of the second of the two trailing vortices may become relevant before the dominance of the term containing $x^{-1}\log x$ in (4.10) is established. The terms in (4.10) containing C_0^2 are associated with the induced drag on the wing, whereas the last term originates, in part at least, in the profile drag. It appears therefore that the contribution to the axial velocity defect from the conversion of the azimuthal motion will be dominant, when the vortex core has increased in diameter to the limit set by the presence of the second trailing vortex, only if the induced drag on the wing is a large fraction of the total drag; we return to this condition in the next section.

Newman (1959) has investigated the flow far downstream in a viscous vortex on the assumption that w and $u-U$ are both of small magnitude by comparison with U, and has argued that in these circumstances the term on the right-hand side of (4.4) may be neglected since it is of the second degree in w; the axial and azimuthal motions then decay independently and only the last term on the right-hand side of (4.10) appears in the asymptotic expression for the axial velocity defect. This is not a self-consistent procedure, because with independent decay the maximum values of $|U-u|$ and $|w|$ diminish as x^{-1} and $x^{-\frac{1}{2}}$ respectively and the term on the right-hand side of (4.4) or (4.7) is asymptotically of the same order in x as the other terms in these equations. Moreover, it is not even safe to argue that the term on the right-hand side of (4.4) or (4.7), although of the same order in x, is in some circumstances numerically much smaller than other terms and thus negligible, because, as we have seen, this term has a cumulative effect on the axial velocity defect and causes it to diminish as $x^{-1}\log x$ rather than as x^{-1}. The correct criterion for the axial and azimuthal motions to decay independently is that the terms containing C_0^2 in (4.10) should be negligible compared with the last term at all values of x for which it is possible to consider one trailing vortex in isolation, and this, as remarked above, is equivalent to the condition that the induced drag on the wing should be a negligible fraction of the total drag.

5. The drag associated with the core of a trailing vortex

It is well known that the relation between the drag D on a body held in a uniform stream and the axial velocity u far downstream in the associated axisymmetric wake (without swirl) is

$$\frac{D}{\rho} = U\int_0^\infty (U-u)\,2\pi r\,dr. \tag{5.1}$$

It would be interesting to know if there is an analogous relation for a trailing line vortex, which would provide a connexion between the velocity distribution determined in the preceding section and the origin of the vortex.

A single trailing vortex cannot exist wholly in isolation, and it is desirable in the first instance to consider the whole wing and vortex system, which will be supposed to consist, far downstream, of two similar vortices with different senses of rotation, one from each side of the wing. The momentum integral theorem will be used, with a control surface enclosing the wing in the form of a right cylinder with generators parallel to the x-axis and of which A is the area of each end face. The upstream end face and the curved surface of the cylinder are both at a large distance from the wing, so that conditions there are approximately as in the free stream. Then in the usual way we find

$$x\text{-momentum flux outwards across curved surface} = \rho U \int (U-u)\,dA,$$

$$x\text{-momentum flux outwards across end faces} = \rho \int (u^2 - U^2)\,dA,$$

$$\text{resultant normal force on end faces} = \int (p_0 - p)\,dA,$$

where u and p are the (x-component of the) velocity and pressure at the downstream end face distance x from the wing, and the small viscous stress at the control surface has been neglected. The drag D on the wing is thus given by

$$\frac{D}{\rho} = \int \left\{ \frac{p_0 - p}{\rho} + u(U-u) \right\} dA. \tag{5.2}$$

Following the procedure for a wake without swirl,† we now choose x to be large, so as to allow the use of approximations to p in (5.2). At positions outside the vortex cores the total head loss ΔH is negligible and

$$p = p_0 + \tfrac{1}{2}\rho U^2 - \tfrac{1}{2}\rho(u^2 + v^2 + w^2),$$

where v and w are orthogonal components of the velocity in the lateral plane; the integrand in (5.2) is then

$$\tfrac{1}{2}(v^2 + w^2) - \tfrac{1}{2}(U-u)^2. \tag{5.3}$$

In the case of a wake without swirl the dominant contribution to the irrotational flow outside the wake, and far from the body, is made by the source-like motion which compensates the axial in-flow in the wake, and it is evident that the integral of (5.3) over the part of the lateral plane outside the wake then tends to zero as $x \to \infty$. In the case of a trailing vortex, the axial inflow $\int (U-u)\,dA$ across a lateral plane distance x_0 downstream does not tend to a constant as $x_0 \to \infty$, but goes on increasing as $\log x_0$ (see (4.9)). The corresponding contribution to the irrotational flow outside the vortex core is effectively that due to sources distributed along the x-axis with line density proportional to x_0^{-1}, and it may be shown that, so far as the effect of these sources is concerned, the integral of (5.3) over the part of the lateral plane outside the vortex core converges and tends to zero as $x \to \infty$. We are therefore free to include in (5.3) only the irrotational motion associated with the axial vorticity in the core.

† See *Applied Hydro- and Aero-mechanics*, by Prandtl and Tietjens (McGraw-Hill 1934), page 125.

At distance r from the centre of one trailing vortex greater than the core radius but small compared with the distance between the two trailing vortices, we have

$$\tfrac{1}{2}(v^2+w^2)-\tfrac{1}{2}(U-u)^2 \approx C_0^2/2r^2.$$

It follows that the integral representing the drag associated with an isolated trailing vortex is logarithmically divergent in the outer field. This divergence reflects the fact that the kinetic energy of the motion in infinite fluid due to a single line vortex with non-zero circulation round it is not finite, and it can also be linked with the variation of the induced vertical velocity at a wing of large span as the reciprocal of the distance from the near wing-tip. However, the divergence is in no way dependent on the structure of the core of the trailing vortex, and it is useful to define the 'drag associated with the core of a trailing vortex' as

$$\frac{D_c}{\rho} = \lim_{R\to\infty}\left[2\pi\int_0^R\left\{\frac{p_0-p}{\rho}+u(U-u)\right\}r\,dr-\pi C_0^2\log\frac{RU}{\nu}\right], \tag{5.4}$$

where p and u now refer to the flow field of an isolated axisymmetric trailing vortex, far downstream, and ν/U has again been used as a convenient reference length in the logarithm. D_c differs from half the total drag on the wing by an amount which depends only on C_0 and the distance between the two trailing vortices far downstream. In the case of two trailing vortices distance s apart far downstream, it may be shown quite readily, from an evaluation of the kinetic energy of the motion in the lateral plane, that the total drag D is

$$D = 2D_c+2\pi\rho C_0^2\log\frac{sU}{\nu}. \tag{5.5}$$

Far downstream the trailing vortex is approximately cylindrical, and the pressure in the core is determined by the balance with centrifugal force, as represented by (2.7). Substitution for p in (5.4) and two integrations by parts then gives

$$\frac{D_c}{\rho} = \tfrac{1}{2}\pi C_0^2+2\pi\int_0^\infty\left\{U(U-u)r-\frac{1}{2}\frac{\partial C^2}{\partial r}\log\frac{rU}{\nu}\right\}dr, \tag{5.6}$$

in which, consistent with the above remarks about the effect of in-flow in the core, $u(U-u)$ has been approximated by $U(U-u)$ and the integrand can be taken as zero outside the vortex core. This is the required generalization of (5.1).

We are now in a position to relate the arbitrary constant L occurring in the similarity solution (4.10) to the more significant parameter D_c. On substituting in (5.6) for $U-u$ from (4.10) and recalling that C is a function of η alone, we find

$$\begin{aligned}\frac{D_c}{\rho} &= \tfrac{1}{2}\pi C_0^2\left\{1+\log\frac{xU}{\nu}\int_0^\infty Q_1\,d\eta-\int_0^\infty Q_2\,d\eta\right.\\ &\qquad\left.-\int_0^\infty\frac{d(C/C_0)^2}{d\eta}\left(\log\eta+\log\frac{4xU}{\nu}\right)d\eta\right\}+\tfrac{1}{2}\pi LU^2,\\ &= \tfrac{1}{2}\pi C_0^2\left\{1-\log 4-\int_0^\infty Q_2\,d\eta-\int_0^\infty\frac{d(C/C_0)^2}{d\eta}\log\eta\,d\eta\right\}+\tfrac{1}{2}\pi LU^2, \end{aligned}\tag{5.7}$$

in which Q_2 and C/C_0 are given explicitly by (4.5) and (5.15). Thus

$$\frac{D_c}{\rho} = -\tfrac{1}{2}\pi\alpha C_0^2 + \tfrac{1}{2}\pi L U^2, \tag{5.8}$$

where α is a positive number not far from unity. The similarity solution (4.10) can now be rewritten as

$$u = U - \frac{1}{8\nu x} e^{-\eta} \left\{ \frac{2}{\pi} \frac{D_c}{\rho} + C_0^2 \left(\alpha + \log \frac{xU}{\nu} \right) \right\} + \frac{C_0^2}{8\nu x} Q_2(\eta). \tag{5.9}$$

If the trailing vortex system from a wing on which the total drag is D consists of two vortices with centres distance s apart, we can use (5.5) to write the similarity solution in the further alternative form

$$u = U - \frac{1}{8\nu x} e^{-\eta} \left\{ \frac{1}{\pi} \frac{D}{\rho} + C_0^2 \left(\alpha + \log \frac{xU}{\nu} - 2 \log \frac{sU}{\nu} \right) \right\} + \frac{C_0^2}{8\nu x} Q_2(\eta). \tag{5.10}$$

The maximum value of x at which one trailing vortex can be considered in isolation occurs when the core diameter is of order s, i.e. when $Us^2/\nu x$ is of order unity. At this value of x the terms containing C_0^2 in (5.10) may be neglected provided

$$\rho C_0^2 \ll D;$$

this is a more specific form of the condition, mentioned in the preceding section, for the axial and azimuthal motions to decay independently.

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